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A relationship between scalar Green functions on hyperbolic and Euclidean Rindler spaces

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Abstract
We derive a formula connecting in any dimension the Green function on the $(D + 1)$-dimensional Euclidean Rindler space and the one for a minimally coupled scalar field with a mass m in the D-dimensional hyperbolic space. The relation takes a simple form in the momentum space where the Green functions are equal at the momenta $(p_0, \mathbf{p})$ for Rindler and $(m, \hat{\mathbf{p}})$ for hyperbolic space with a simple additive relation between the squares of the mass and the momenta. The formula has applications to finite temperature Green functions, Green functions on the cone and on the (compactified) Milne spacetime. Analytic continuations and interacting quantum fields are briefly discussed.

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1. Introduction

The Green functions define quantum field theory in the Minkowski space as well as in the curved space. In the Minkowski space the relation is one to one if we impose the restrictions of locality and Poincare invariance. In the curved space the Green function is not unique. This is a consequence of the non-uniqueness of the physical vacuum [1, 2]. There is less ambiguity for the Green functions on the Riemannian manifolds (instead of the physical pseudo-Riemannian ones). However, in the Euclidean approach [3–5] we can construct only a subset of quantum field theories admissible on the pseudo-Riemannian manifolds. The Euclidean version on a manifold can have more than one analytic continuation. The hyperbolic space can be continued analytically either to de Sitter space or to anti-de Sitter space. In the case of the de Sitter and anti-de Sitter spaces the relation between the Riemannian and pseudo-Riemannian approaches is well understood (the Euclidean approach is distinguishing the 'Euclidean vacuum' also known as the 'Bunch–Davis vacuum' [6]).

In this paper we discuss Green functions on the Euclidean version of the Rindler space and on the hyperbolic space. The Euclidean Rindler space continues analytically either to the conventional Rindler space of an accelerated observer [7] or to the Milne space [2] which plays an important role in the 'ekpyrotic' scenario [8]. de Sitter and anti-de Sitter spaces can be considered as asymptotic solutions (near the singularity) of gravity and string models (black branes [9], see also [10]). The Rindler space describes near the horizon geometry of spacetime. We find that after a Fourier transformation in time the massless Green function in the Rindler space is equal to the Green function in the hyperbolic space for a quantum field with a mass m (related to p_0). The relation can be extended to the Rindler Green function with mass m_{R}, but in such a case we have to take the Fourier transform in spatial coordinates as well. Then, the Fourier transforms of both Green functions are equal for momenta $(p_0, \mathbf{p})$ (Rindler) and $(m, \hat{\mathbf{p}})$ (hyperbolic).

The Green functions on the Rindler space have been discussed by many authors [1, 6, 11–13]. A requirement of the (twisted) periodic conditions relates Green functions on the conical manifold [14–16] (a solution for the cosmic string [17, 18]) to the Rindler Green functions and Rindler quantum fields at finite temperature (then there is no twist). These Green functions have been studied by many authors (mainly in four dimensions) by means of an eigenfunction expansion [19–22]. It seems however that the simple relations derived in this paper have not been known before. The Green functions and quantum fields on de Sitter and anti-de Sitter spaces have been discussed in many physics papers [10, 23–31] as well as in mathematics [32–34].

2. Green functions

We are interested in metrics with a bifurcate Killing horizon in $(D + 1)$-dimensional pseudo-Riemannian manifold $\mathcal{N}$. This notion is defined (see [35] for details) by a Killing vector vanishing on a $(D - 1)$-dimensional surface. The bifurcate Killing horizon locally divides the spacetime into four wedges as the boost Killing field does it in the case of the Minkowski space [7, 35]. We take as our starting point the Euclidean version of the Rindler approximation of the metric on $\mathcal{N}$:

$$\mathrm{d}s^2 \equiv g_{AB}\,\mathrm{d}x^A\,\mathrm{d}x^B = y^2(\mathrm{d}x^0)^2 + \mathrm{d}y^2 + \mathrm{d}\mathbf{x}^2. \tag{1}$$

An analytic continuation $x^0 \to \mathrm{i}x^0$ transforms the metric (1) back into the pseudo-Riemannian Rindler metric. The analytic continuation of both $x^0 \to \mathrm{i}x^0$ and $y \to \mathrm{i}t$ transforms the metric (1) into the Milne metric [2].

Let

$$\triangle_N = \frac{1}{\sqrt{g}}\partial_A g^{AB}\sqrt{g}\partial_B \tag{2}$$

be the Laplace–Beltrami operator on $\mathcal{N}$ (here $g = \det(g_{AB})$).

We are interested in the calculation of the Green functions for a minimal coupling of the scalar field

$$(-\triangle_N + m^2)\mathcal{G}_N^m = \frac{1}{\sqrt{g}}\delta. \tag{3}$$

A solution of equation (3) can be expressed by the fundamental solution of the diffusion equation

$$\partial_\tau P = \tfrac{1}{2}\triangle_N P \tag{4}$$

with the initial condition $P_0(x, x') = g^{-\frac{1}{2}}\delta(x - x')$. Then

$$\mathcal{G}_N^m = \frac{1}{2}\int_0^\infty \mathrm{d}\tau\,\exp\left(-\frac{1}{2}m^2\tau\right)P_\tau. \tag{5}$$

In order to prove equation (5) we multiply equation (4) by $\exp\left(-\frac{1}{2}m^2\tau\right)$ and integrate both sides over τ applying the initial condition for P_τ.

In the metric (1) we have

$$\triangle_N = y^{-2}\partial_0^2 + y^{-1}\partial_y y\partial_y + \triangle, \tag{6}$$

where $\triangle$ is the Laplacian on R^n where $n = D-1$. After an exponential change of coordinates

$$y = \exp u \tag{7}$$

equation (3) reads (we denote the mass of the Rindler field by m_{R})

$$\left(-\partial_0^2 - \partial_u^2 + \exp(2u)\left(-\triangle + m_{\mathrm{R}}^2\right)\right)\mathcal{G}_R^m = \delta(x_0 - x_0')\delta(u-u')\delta(\mathbf{x}-\mathbf{x}'). \tag{8}$$

The metric (1) by means of the conformal transformation is related to the metric $(\mathrm{d}x^0)^2 + \mathrm{d}s_H^2$ on $R \times \mathcal{H}_D$, where

$$\mathrm{d}s_H^2 = y^{-2}(\mathrm{d}y^2 + \mathrm{d}\mathbf{x}^2) \tag{9}$$

(with $\mathbf{x} \in R^{D-1}$) is the Riemannian metric (the Poincare metric) on the hyperbolic space $\mathcal{H}_D = SO(D,1)/SO(D)$.

The Laplace–Beltrami operator (2) for the hyperbolic space reads

$$\triangle_H = y^2\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n^2}\right) - (n-1)y\frac{\partial}{\partial y}. \tag{10}$$

Let

$$\mathcal{G}_H^m(X, X') = y^{\frac{n}{2}}y'^{\frac{n}{2}}\hat{\mathcal{G}}_H^m(X, X'), \tag{11}$$

where $X = (y, \mathbf{x})$. Then, $\hat{\mathcal{G}}_H^m$ satisfies the equation

$$\left(-y\partial_y y\partial_y - y^2\triangle + \frac{n^2}{4} + m^2\right)\hat{\mathcal{G}}_H^m = y\delta(X - X'). \tag{12}$$

In coordinates (7), equation (12) reads

$$\left(-\partial_u^2 - \exp(2u)\triangle + m^2 + \frac{n^2}{4}\right)\hat{\mathcal{G}}_H^m = \delta(u-u')\delta(\mathbf{x}-\mathbf{x}'). \tag{13}$$

We also consider the heat kernel for the hyperbolic space, i.e., the fundamental solution of the diffusion equation

$$\partial_\tau P_\tau^H = \tfrac{1}{2}\triangle_H P_\tau^H, \tag{14}$$

with the initial condition $P_0(X, X') = \frac{1}{\sqrt{g}}\delta(X - X')$.

If we have the fundamental solution (14), then we can solve the equation for the Green function

$$(-\triangle_H + m^2)\mathcal{G}_H^m = \frac{1}{\sqrt{g}}\delta. \tag{15}$$

Let us define

$$P_\tau^H(X, X') = y^{\frac{n}{2}}y'^{\frac{n}{2}}\hat{P}_\tau(X, X'). \tag{16}$$

Then, $\hat{P}$ satisfies

$$-\partial_\tau\hat{P}_\tau = \frac{1}{2}\left(-\partial_u^2 - \exp(2u)\triangle + m^2 + \frac{n^2}{4}\right)\hat{P}_\tau, \tag{17}$$

with the initial condition

$$\hat{P}_0(X, X') = y\delta(X - X').$$

Therefore $\hat{\mathcal{G}}$ defined in equation (11) is expressed by $\hat{P}$:

$$\hat{\mathcal{G}}_H^m = \int_0^\infty d\tau \exp\left(-\frac{1}{2}m^2\tau\right)\hat{P}_\tau. \tag{18}$$

We are prepared now to show a correspondence between the Green function for the Rindler space and the Green function on the hyperbolic space. The correspondence could be guessed on the basis of the conformal equivalence mentioned above equation (9) (see a discussion of conformal invariance in [36–38]). In order to prove the relationship between the Green functions let us consider the Fourier transform

$$\mathcal{G}_R^m(x_0, u, \mathbf{x}; x_0', u', \mathbf{x}') = \frac{1}{2\pi}\int dp_0 \exp(ip_0(x_0 - x_0'))\tilde{\mathcal{G}}_R^m(p_0, u, u'; |\mathbf{x}-\mathbf{x}'|). \tag{19}$$

It follows from equations (8) and (13) that at $m_R = 0$:

$$\tilde{\mathcal{G}}_R^0(p_0, u, u'; |\mathbf{x}-\mathbf{x}'|) = \hat{\mathcal{G}}_H^m(u, u'; |\mathbf{x}-\mathbf{x}'|) \tag{20}$$

if

$$p_0^2 = m^2 + \frac{n^2}{4} \tag{21}$$

(in this way the mass in the hyperbolic space acquires the meaning of a momentum in the $D+1$ dimension). Applying equations (8), (13) and (18), we obtain

$$\tilde{\mathcal{G}}_R^0(p_0, u, u'; |\mathbf{x}-\mathbf{x}'|) = \int_0^\infty d\tau \exp\left(-\frac{\tau}{2}p_0^2 + \frac{\tau}{8}n^2\right)\hat{P}_\tau(y, \mathbf{x}; y', \mathbf{x}') \tag{22}$$

or

$$\mathcal{G}_R^0(x_0, u, \mathbf{x}; x_0', u', \mathbf{x}') = \int_0^\infty d\tau (2\pi\tau)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\tau}(x_0 - x_0')^2 + \frac{\tau}{8}n^2\right)\hat{P}_\tau(y, \mathbf{x}; y', \mathbf{x}'). \tag{23}$$

The relation can be extended to $m_R \neq 0$. Let us denote the Fourier transform of $\mathcal{G}(\mathbf{x})$ by $\tilde{\mathcal{G}}(\mathbf{p})$. The Green function $\tilde{\mathcal{G}}(\mathbf{p})$ depends only on $|\mathbf{p}|$. Let us denote the Fourier transform of the massive Rindler Green function $\tilde{\mathcal{G}}_R^{m_R}(p_0, u, u'; |\mathbf{x}-\mathbf{x}'|)$ in the $\mathbf{x}-\mathbf{x}'$ variable by $\tilde{\mathcal{G}}_R^{m_R}(p_0, u, u'; |\mathbf{p}|)$. Then

$$\tilde{\mathcal{G}}_R^{m_R}(p_0, u, u'; |\mathbf{p}|) = \tilde{\hat{\mathcal{G}}}_H^m(u, u'; |\hat{\mathbf{p}}|) = \mathcal{G}_Q(|\hat{\mathbf{p}}|; u, u'), \tag{24}$$

if in addition to equation (21) (expressing m by p_0) the following relation is satisfied:

$$m_R^2 + \mathbf{p}^2 \equiv \omega(\mathbf{p})^2 = \hat{\mathbf{p}}^2 \tag{25}$$

(the momentum in the hyperbolic space acquires the meaning of the energy in the Rindler space). On the rhs of equation (24) $\mathcal{G}_Q(|\hat{\mathbf{p}}|)$ is the integral kernel $\mathcal{A}_Q^{-1}$ of the quantum mechanical Hamiltonian $\mathcal{A}_Q$ (with an exponential potential)

$$\mathcal{A}_Q = -\partial_u^2 + \hat{\mathbf{p}}^2 \exp(2u) + m^2 + \frac{n^2}{4}. \tag{26}$$

Formula (24) can be rewritten in the configuration space. For this purpose we express $\hat{\mathbf{p}}$ on the rhs of equation (24) by $\omega(\mathbf{p})$ from equation (25). Then, the Fourier transform of equation (24) can be expressed by means of a kernel K_{m_R} relating the massive Green function to the massless one

$$\mathcal{G}_R^{m_R}(x_0, y, \mathbf{x}; x_0', y', \mathbf{x}') = \int d\hat{\mathbf{x}}\, K_{m_R}(\mathbf{x}-\mathbf{x}', \hat{\mathbf{x}})\mathcal{G}_R^0(x_0, y, \hat{\mathbf{x}}; x_0', y', \mathbf{0}), \tag{27}$$

where

$$K_{m_{\mathrm{R}}}(\mathbf{x}-\mathbf{x}',\hat{\mathbf{x}}) = (A(n-1))^{-1}(2\pi)^{-n} \times \int \mathrm{d}\mathbf{p}\,\mathrm{d}\hat{\mathbf{p}}\exp(-\mathrm{i}\hat{\mathbf{p}}\hat{\mathbf{x}} + \mathrm{i}\mathbf{p}(\mathbf{x}-\mathbf{x}'))\delta(|\hat{\mathbf{p}}| - \omega(\mathbf{p}))\omega(\mathbf{p})^{-n+1} \tag{28}$$

and $A(n-1)$ is the area of the $(n-1)$-dimensional sphere of radius 1. In order to prove equation (27) we take the Fourier transform of equation (27) in $\mathbf{x}$. Then the remaining integral over $\hat{\mathbf{p}}$ can be performed in spherical coordinates leading to formula (24).

3. Integral representation

The heat kernel P_τ (14) on the hyperbolic space has been calculated by many authors (see [39] for a review; we have done the calculations by means of a probabilistic method [40] and our results agree with those of [41]). It is a function of the Riemannian distance σ:

$$\cosh\sigma \equiv z = 1 + (2yy')^{-1}((\mathbf{x}-\mathbf{x}')^2 + (y-y')^2).$$

We have for odd dimensions $D = n+1 = 2k+3$ $(k = 0, 1, \ldots)$ (where $P_\tau(y,\mathbf{x};y',\mathbf{x}') \equiv p_\tau(\sigma)$):

$$p_\tau^{(k+1)}(\sigma) = (-2\pi)^{-k}\exp\left(-\frac{n^2}{8}\tau + \frac{1}{2}\tau\right)\left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k p_\tau^{(1)}(\sigma), \tag{29}$$

with

$$p_\tau^{(1)}(\sigma) = (2\pi\tau)^{-\frac{3}{2}}\sigma(\sinh\sigma)^{-1}\exp\left(-\frac{\tau}{2} - \frac{\sigma^2}{2\tau}\right), \tag{30}$$

where $(\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma} = \frac{\mathrm{d}}{\mathrm{d}z}$.

In even dimensions $D = n+1 = 2k+2$:

$$p_\tau^{(k)}(\sigma) = \exp\left(-\frac{n^2\tau}{8} + \frac{\tau}{8}\right)(-2\pi)^{-k}\left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k p_\tau^{(0)}(\sigma), \tag{31}$$

where [42]

$$p_\tau^{(0)}(\sigma) = \exp\left(-\frac{\tau}{8}\right)\sqrt{2}(2\pi\tau)^{-\frac{3}{2}}\int_\sigma^\infty(\cosh r - \cosh\sigma)^{-\frac{1}{2}} r\exp\left(-\frac{r^2}{2\tau}\right)\mathrm{d}r. \tag{32}$$

Then the Green functions for the hyperbolic space read $(n = 2k+2, k = 0, 1, \ldots)$,

$$\mathcal{G}_H^m(y,\mathbf{x};y',\mathbf{x}')_{2k+2} = (-2\pi)^{-k}\left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k(\sinh\sigma)^{-1}\exp(-\nu\sigma), \tag{33}$$

where

$$\nu = \sqrt{\frac{n^2}{4} + m^2} \tag{34}$$

and for an odd n,

$$\begin{aligned}\mathcal{G}_H^m(y,\mathbf{x};y',\mathbf{x}')_{2k+1} &= 2\sqrt{2}(2\pi)^{-\frac{3}{2}}(-2\pi)^{-k}\left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k\int_\sigma^\infty(\cosh r - \cosh\sigma)^{-\frac{1}{2}}\exp(-\nu r)\,\mathrm{d}r\\ &= 2(2\pi)^{-\frac{3}{2}}(-2\pi)^{-k}\left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k Q_{\nu-\frac{1}{2}}(\cosh\sigma),\end{aligned} \tag{35}$$

where the Legendre function Q has the integral representation [43]

$$Q_\alpha(z) = \int_\sigma^\infty (2\cosh r - 2z)^{-\frac{1}{2}} \exp\left(-\frac{(2\alpha+1)r}{2}\right) \mathrm{d}r.$$

The Fourier transform in x_0 of the massless Rindler Green function is equal to the hyperbolic Green function with (see equations (20), (21) and (34))

$$\nu = |p_0|$$

in equations (33) and (35). Explicitly, for an even case $n = 2k + 2$ ($D = n + 2$) we have the simple formula

$$\tilde{\mathcal{G}}_R^0(p_0, y, \mathbf{x}; y', \mathbf{x}')_{2k+2} = (-2\pi)^{-k} y^{-k-1} y'^{-k-1} \times \left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k (\sinh\sigma)^{-1} \exp(-|p_0|\sigma). \quad (36)$$

The massless Green function in ($D + 1 = 2k + 4$)-dimensional Rindler space can be obtained either from equation (36) by means of the Fourier transform in p_0 or from equation (23) by a calculation of the τ-integral

$$\mathcal{G}_R^0(x_0 - x_0', y, y'; \sigma)_{2k+2} = (-2\pi)^{-k-2} y^{-k-1} y'^{-k-1} \times \left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k \sigma(\sinh\sigma)^{-1}\left(\sigma^2 + (x_0 - x_0')^2\right)^{-1} \quad (37)$$

(for $k = 0$ the formula has been derived in [12, 14, 15, 44]).

In even dimensions $D = 2k + 2$ the formula for the Rindler Green function is more complicated. From equations (23) and (32), we obtain

$$\mathcal{G}_R^0(x_0 - x_0', y, y'; \sigma)_{2k+1} = -(-2\pi)^{-k-1} y^{-k-\frac{1}{2}} y'^{-k-\frac{1}{2}} \times \left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k \int_\sigma^\infty (\cosh r - \cosh\sigma)^{-\frac{1}{2}} r (r^2 + (x_0 - x_0')^2)^{-1} \mathrm{d}r. \quad (38)$$

We can extend the formulae to quantum field theory at finite temperature and to a construction of Green functions (with a zero twist) on the conical manifolds [14, 22]. The Euclidean Green functions at finite temperature are constructed [45] from $\mathcal{G}$ by an imposition of the periodicity condition in time by means of the method of images. Applying the formula

$$\sum_n \left(\sigma^2 + (x_0 - x_0' + n\beta)^2\right)^{-1} = \pi(2\beta\sigma)^{-1}\left(\coth\left(\frac{\pi}{\beta}(\sigma + \mathrm{i}(x_0 - x_0'))\right) + \coth\left(\frac{\pi}{\beta}(\sigma - \mathrm{i}(x_0 - x_0'))\right)\right),$$

we obtain (in four dimensions the formula has been derived by [13–15, 46])

$$\mathcal{G}_{2k+2}^0(x_0 - x_0', y, y'; \sigma)_\beta = (-2\pi)^{-k-2} y^{-k-1} y'^{-k-1} \times \left((\sinh\sigma)^{-1}\frac{\mathrm{d}}{\mathrm{d}\sigma}\right)^k \sinh\left(\frac{2\pi}{\beta}\sigma\right) \times \pi(2\beta\sinh\sigma)^{-1}\left(\cosh\left(\frac{2\pi}{\beta}\sigma\right) - \cos\left(\frac{2\pi}{\beta}(x_0 - x_0')\right)\right)^{-1} \quad (39)$$

and

$$\mathcal{G}^0_{2k+1}(x_0 - x'_0, y, y'; \sigma)_\beta = -(-2\pi)^{-k-1} y^{-k-\frac{1}{2}} y'^{-k-\frac{1}{2}} \left((\sinh\sigma)^{-1} \frac{\mathrm{d}}{\mathrm{d}\sigma} \right)^k$$
$$\times \int_\sigma^\infty \mathrm{d}r \sinh\left(\frac{2\pi}{\beta} r\right) (\cosh r - \cosh\sigma)^{-\frac{1}{2}} \pi (2\beta)^{-1}$$
$$\times \left(\cosh\left(\frac{2\pi}{\beta} r\right) - \cos\left(\frac{2\pi}{\beta}(x_0 - x'_0)\right) \right)^{-1}. \tag{40}$$

The Green functions for the massive Rindler model at finite temperature (and for massive fields on the conical manifold) are defined in equation (27). From equations (39) and (40) it is easy to see that the massless Green functions at $\beta = 2\pi$ coincide with the ones of the free quantum field on the Euclidean space as they should because the metric (1) coincides with the flat metric in polar coordinates when x_0 is periodic with the period 2π. The momentum of the finite temperature Rindler Green functions is discrete $p_0 = 2\pi l \beta^{-1}$ in equation (36), where $l = 0, \pm 1, \ldots$. Then in equation (35)

$$\nu = 2\pi |l| \beta^{-1}.$$

For $\beta = 4\pi |l| (2k+1)^{-1}$, where k is a natural number, the Legendre functions are expressed by elementary functions of z as in the case of the massless Green function on the D-dimensional hyperbolic space.

4. An outlook:analytic continuation and quantum fields

We can discuss now analytic continuations of the Green functions as functions of the Riemannian metric (1). The standard approach starts with a pseudo-Riemannian metric [1, 2]. For a class of manifolds we can construct quantum fields and calculate their Green functions. These Green functions can be continued analytically to the Riemannian metric (the Euclidean region). Starting from the Riemannian metric we encounter some problems with an analytic continuation, as discussed e.g. in [3]. There may be no analytic continuation to a quantum field theory on a curved background or the analytic continuation may be not unique. There is no difficulty in the case of free fields defined on static spacetimes (the problem of an analytic continuation has been solved in [3]). The interacting fields in the flat spacetime have an analytic continuation if their Euclidean version is Osterwalder–Schrader (OS) positive (then the Minkowski version is Wightman positive). It has been noted some years ago (see [4, 5] and references therein) that the reflection positivity of Green functions defined on Riemannian manifolds allows us to construct quantum fields on some pseudo-Riemannian manifolds although their physical meaning (in particular the particle interpretation) may be obscure.

We discuss here solely analytic continuations of the Euclidean Rindler model. First, we can continue, $x^0 \to \mathrm{i}x^0$. Then, we obtain the usual Rindler space. The continuation of Green functions can be performed explicitly using equations (37) and (38) (and (27) for the massive case). It can be seen directly from the Green functions (37) and (38) that they are OS positive with respect to the reflection $x_0 \to -x_0$ (this is so because $(a^2 + (x_0 - x'_0)^2)^{-1}$ is OS positive). The analytic continuation could also be performed by means of the operator formalism as in [3] (because the metric in the Laplace–Beltrami operator (6) is time independent). Next, we can continue analytically $x_1 \to \mathrm{i}x_1$ (or equivalently any x_k with $k > 1$). In such a case we obtain a manifold which is conformal to $R \times \mathrm{AdS}$ (or to $S^1 \times \mathrm{AdS}$ for the periodic Green functions (39) and (40)). Its Green function can be obtained explicitly in the even case (37). The odd case (38) is more complicated for negative (time-like) $z - 1 = \cosh\sigma - 1$. In such a case an analytic continuation of equation (38) is needed. We may use a definition of the Legendre function in terms of the hypergeometric function for this purpose (see [43]) or consider the analytic continuation of the Legendre functions directly from the integral representation [47]. The reflection positivity of Euclidean Green functions (37) and (38) and the Wightman positivity of quantum fields (as its consequence) are not obvious but shown in detail in [5, 31].

The third possibility is to continue analytically $x_0 \to \mathrm{i}x_0$ simultaneously with $y \to \mathrm{i}t$. In such a case we obtain the Milne space [2, 48]. An analytic continuation of equation (37) gives the Green function for the Milne space in the odd case. Formula (38) for the even case needs an analytic continuation (σ can be imaginary) by means of the hypergeometric function. Analytic continuation of equations (39) and (40) gives the formula for the Green functions of the compactified Milne space discussed in [38, 50, 51]. However, it is not clear whether the analytically continued Green functions satisfy the Wightman positivity (i.e., if the model defines quantum fields).

There is another way to construct an analytic continuation using the expansion in a complete set of solutions of the Klein–Gordon equation. Then, the Euclidean Rindler Green function is expressed in the form [1, 21, 48, 49]

$$\begin{aligned} \mathcal{G}_R^0(x_0, y, \mathbf{x}; x_0', y', \mathbf{x}') &= 8(2\pi)^{-D-1} \int \mathrm{d}p_1\, \mathrm{d}\mathbf{p}\, \sinh(\pi|p_1|) \\ &\quad \times \exp(-|p_1||x_0 - x_0'| - \mathrm{i}\mathbf{p}(\mathbf{x} - \mathbf{x}')) K_{ip_1}\big(|\mathbf{p}|\sqrt{y^2}\big)\overline{K_{ip_1}\big(|\mathbf{p}|\sqrt{y'^2}\big)}, \end{aligned} \tag{41}$$

where $K_\nu(y) = K_{-\nu}(y)$ is the modified Bessel function of the third kind of order ν [43] vanishing at infinity; the square root and the complex conjugation indicate the path of an analytic continuation from $y > 0$ to y on the complex plane. The Green function (41) is reflection positive with respect to the reflection $\theta x_0 = -x_0, \theta y = -y$ if we extend it from the Rindler wedge to $y \leqslant 0$ by equation (41). It is Wightman positive if the analytic continuation $y \to \mathrm{i}t$ in equation (41) is defined by the formula

$$\exp(-\mathrm{i}p_1(x_0 - x_0') - \mathrm{i}\mathbf{p}(\mathbf{x} - \mathbf{x}'))K_{ip_1}(\mathrm{i}|\mathbf{p}|t)\overline{K_{ip_1}(\mathrm{i}|\mathbf{p}|t')}. \tag{42}$$

The form of the two-point function resulting from the analytic continuation (42) coincides with the one which we obtain if we expand the Rindler quantum field in creation and annihilation operators (see e.g. [1]) and subsequently continue analytically the modes. We may have difficulties with physical interpretation of quantum fields in a time-dependent metric. We note however that in the Milne case at least in the limit $t = \exp(u) \to 0$ ($u \to -\infty$ in equations (7) and (8)) the quantum field splits into the positive and negative frequency parts (as $K_{ip_1}(\mathrm{i}|\mathbf{p}|t) \simeq \exp(\mathrm{i}p_1 u)$ if $u \to -\infty$; for some other quantum fields in a time-dependent metric see [24, 52]). However, we do not know whether the procedure of the analytic continuation suggested here for the Milne model is unique. The resulting two-point function may be different from the one which would have come from an analytic continuation of equations (37) and (38). This problem is still under investigation [53].

As a next step an interaction could be defined which determines the S-matrix in terms of the propagators. The relation between Rindler and (anti) de Sitter propagators (they are equal in the momentum space, equation (20)) indicates that the two approximations for a near horizon geometry may lead to equivalent physical results concerning scattering processes (however we should keep in mind that when calculating with hyperbolic propagators we must still integrate over the mass).

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